

LIMITS OF POLYNOMIALS WHOSE ZEROS LIE IN A RADIAL SET

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1. Introduction. Let R be an unbounded closed set in the complex plane. The polynomials whose zeros belong to R will be called R -polynomials. We say that a sequence of functions is U -convergent when it converges everywhere in the plane, and uniformly on every bounded set. The object of study is the class $C(R)$ of all functions not identically zero which can be obtained as U -limits of R -polynomials. It is clear that $C(R)$ consists of entire functions whose zeros belong to R . A set R such that $C(R)$ consists of *all* entire functions not identically zero whose zeros belong to R will be called *regular*. Descriptions of $C(R)$ for a number of special "singular" sets have been known for a long time; cf. Obrechhoff's monograph [6]. For a survey of more recent work, see [3].

Some years ago the second author set the problem of describing the class $C(R)$ for general sets R . He obtained a number of results in terms of the asymptotic directions of R . It is convenient to define the asymptotic directions as the rays $\arg z = \theta$ for which R contains a sequence of points $\{z_n\}$ which tends to infinity in such a way that $\arg z_n \rightarrow \theta$. The set of the asymptotic directions (rays) $\arg z = \theta$ for R will be denoted by $A(R)$, the set of the corresponding rays $\arg z = j\theta$ by $A^j(R)$ or $A(R^j)$. One can distinguish three kinds of sets R :

(i) R will be called a set of the first kind if no set $A^j(R)$, $j = 1, 2, \dots$, belongs to a half-plane. Sets R of the first kind are regular [1].

Suppose now that we have a set R not of the first kind. For such a set, let k be the smallest positive integer such that $A^k(R)$ belongs to a half-plane.

(ii) R will be called a set of the second kind if the convex hull of $A^k(R)$ is an angle $< \pi$. For a set of the second kind $C(R)$ consists of the entire functions of the form $\exp(cz^k) \cdot g(z)$, where $g(z)$ is an entire function not identically zero of genus $\leq k - 1$ whose zeros belong to R , and $-\bar{c}$ belongs to the convex hull of $A^k(R)$ [1].

(iii) R will be called a set of the third kind if the convex hull of $A^k(R)$ is a half-plane or a line. Some sets of the third kind are regular [1], some are singular; no general results are known. Of the singular sets of the third kind only special sets such as lines, strips and half-planes have been investigated.

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In work of this sort it is convenient to introduce, besides the order and the genus of an entire function $f(z)$, a third number which we shall call its *index*. This is the smallest positive integer q such that $f(z)$ can be written in the form $\exp(cz^q) \cdot g(z)$, where $g(z)$ is entire of genus $\leq q - 1$. The order ρ (index q) of a set R or the class $C(R)$ is defined as the least upper bound of the orders (indices) of the entire functions in $C(R)$. There exist sets of all orders ρ , $1 \leq \rho \leq \infty$. It is known that a set is singular if and only if its order (or index) is finite [2]. For a singular set R the zero free functions can be written in the form

$$A \exp(a_1 z + \dots + a_q z^q) \quad (A \neq 0).$$

Let us denote by K the set of all corresponding points

$$a = (a_1, \dots, a_q) = (\operatorname{Re} a_1, \operatorname{Im} a_1, \dots, \operatorname{Re} a_q, \operatorname{Im} a_q)$$

in real $2q$ -dimensional space. It is known that K is a (closed) convex cone: if $a \in K$ then $\lambda a \in K$ for all real $\lambda \geq 0$, and if $a, b \in K$ then $\frac{1}{2}(a + b) \in K$ [2].

2. Present results. In this paper we obtain a complete description of the class $C(R)$ for the case where R is any (closed) radial set, be it of the first, second or third kind. For a radial set, that is, a set such that if $z \in R$ then also $\lambda z \in R$ for all $\lambda \geq 0$, $A(R)$ coincides with R . A radial set is regular if and only if it is of the first kind, that is, if and only if no power R^j belongs to a half-plane.

For a singular radial set we determine the index in terms of the geometry. The order is equal to the index, and attained by a zero free function. The corresponding cone K is the product of 2-dimensional closed convex cones K_j : $a \in K$ if and only if $a_j \in K_j$, $j = 1, \dots, q$. For $j < \frac{1}{2}q$ the set K_j is the plane, for $\frac{1}{2}q \leq j < q$ the K_j are either the plane, a half-plane, or a line; K_q is always an angle $< \pi$ or a ray. (For certain nonradial sets, such as the half-plane $\operatorname{Re} z \geq 1$, the cone K cannot be decomposed in this manner [1].)

In order to discuss our results for singular radial sets in more detail we have to describe the geometry of R . Set $R = R_1$, let k_1 be the smallest positive integer such that $R_1^{k_1}$ belongs to a half-plane, and let S_1 be the convex hull of $R_1^{k_1}$. If S_1 is an angle $< \pi$ the set R is of the second kind and $C(R)$ is known. If S_1 is a half-plane $\alpha_1 \leq \arg z \leq \alpha_1 + \pi$ or a line $\arg z = \alpha_1$, $\alpha_1 + \pi$ we introduce an auxiliary set R_2 . The set R_2 will consist of those rays $\arg z = \theta$ of R_1 for which $k_1 \theta \equiv \alpha_1 \pmod{\pi}$. That is, R_2 consists of the rays of R_1 whose k_1 th powers fall along the boundary of S_1 .

In general, let k_ν be the smallest integer $> k_{\nu-1}$ ($k_0 = 0$) such that $R_\nu^{k_\nu}$ belongs to a half-plane, and let S_ν be the convex hull of $R_\nu^{k_\nu}$. If S_ν is

an angle $< \pi$ we go no further. If S_ν is a half-plane $\alpha_\nu \leq \arg z \leq \alpha_\nu + \pi$ or a line $\arg z = \alpha_\nu, \alpha_\nu + \pi$ we introduce an auxiliary set $R_{\nu+1}$. The set $R_{\nu+1}$ will consist of those rays $\arg z = \theta$ of R_ν for which $k_\nu \theta \equiv \alpha_\nu \pmod{\pi}$. In other words, $R_{\nu+1}$ consists of the rays of R_ν whose k_ν th powers fall along the boundary of S_ν .

The above process comes to an end in a finite number of steps since $R_1 \supset R_2 \supset \dots$, and $R_{2^{k_1}}$ is just a ray. Let R_ω be the last set R_ν introduced in the construction, and let k_ω be the smallest integer $> k_{\omega-1}$ such that $R_\omega^{k_\omega}$ belongs to a half-plane. Then the convex hull S_ω of $R_\omega^{k_\omega}$ is an angle $< \pi$ or a ray. Note that since $R_\nu^{2^{k_1}}$ is a ray for any $\nu \geq 2$, one has

$$(2.1) \quad k_\omega \leq 2k_1$$

whenever $\omega \geq 2$.

THEOREM 2.1. *Let R be a singular radial set. Then the zero free entire functions in $C(R)$ are the functions of the form*

$$(2.2) \quad A \exp(a_1 z + \dots + a_N z^N) \quad (A \neq 0)$$

where $N = k_\omega$,

$$(2.3) \quad a_{k_\nu} \in -\bar{S}_\nu \quad \text{for } \nu = 1, \dots, \omega,$$

and the other a_j are arbitrary. (Note that by (2.1) all a_j with $j < \frac{1}{2}N$ are arbitrary.)

In (2.3), $-\bar{S}_\nu$ denotes the set of all points $-\bar{z}$ with $z \in S_\nu$.

THEOREM 2.2. *The singular radial set R has index and order equal to $N = k_\omega$. The class $C(R)$ consists of the entire functions which can be represented in the form*

$$(2.4) \quad A z^m \exp(b_1 z + \dots + b_N z^N) \cdot \prod_p (1 - z/z_p) \exp(z/z_p + \dots + z^{N-1}/(N-1)z_p^{N-1})$$

where $A \neq 0$, $z_p \in R$, $\sum_p |z_p|^{-N}$ converges, b_j is arbitrary for $j \neq k_\nu$, $\nu = 1, \dots, \omega$, $b_{k_\omega} \in -\bar{S}_\omega$, and

$$(2.5) \quad b_{k_\nu} + \frac{1}{k_\nu} \sum_p z_p^{-k_\nu} \in -\bar{S}_\nu, \quad \nu = 1, \dots, \omega - 1,$$

in the sense that the series

$$(2.6) \quad \operatorname{Im}(b_{k_\nu} e^{i\alpha_\nu}) + \frac{1}{k_\nu} \sum_p \operatorname{Im}(z_p^{-k_\nu} e^{i\alpha_\nu})$$

converges to a sum

$$(2.7) \quad \sigma_\nu \left\{ \begin{array}{l} \geq 0 \text{ if } S_\nu \text{ is the half-plane } \alpha_\nu \leq \arg z \leq \alpha_\nu + \pi, \\ = 0 \text{ if } S_\nu \text{ is the line } \arg z = \alpha_\nu, \alpha_\nu + \pi. \end{array} \right.$$

A somewhat less precise form of the above results was obtained by the first author in his (unpublished) Ph.D. thesis [4]. The proofs in the present paper are shorter and quite different. They depend on comparison of $C(R)$ with certain related classes of entire functions, a technique which may be useful also in the study of nonradial sets.

3. The auxiliary classes $C_h(R)$. We begin by normalizing the functions in $C(R)$. If $F(z)$ is any entire function not identically zero one can write $F(z) = Az^m f(z)$, with $f(0) = 1$. Furthermore, if $F(z)$ is the U -limit of R -polynomials $F_n(z)$, one can omit suitable factors from the $F_n(z)$ to obtain R -polynomials $f_n(z)$ which are U -convergent to $f(z)$. Thus $f(z) \in C(R)$. We denote the subset of the functions $f(z) \in C(R)$ with $f(0) = 1$ by $C_0(R)$. Every function $f(z) \in C_0(R)$ is the U -limit of normalized R -polynomials

$$(3.1) \quad f_n(z) = \prod_p (1 - z/z_{np}), \quad z_{np} \in R.$$

More generally, if h is any integer ≥ 0 we define $C_h(R)$ as the class of U -limits of finite products of the form

$$(3.2) \quad f_n(z; h) = \prod_p (1 - z/z_{np}) \exp(z/z_{np} + \dots + z^h/hz_{np}^h), \quad z_{np} \in R.$$

Note that for $|z| < \min_p |z_{np}|$,

$$(3.3) \quad f_n(z; h) = \exp(s_{n,h+1}z^{h+1} + s_{n,h+2}z^{h+2} + \dots)$$

where the s_{nj} are given by the formula

$$(3.4) \quad s_{nj} = -\frac{1}{j} \sum_p z_{np}^{-j}, \quad j = 1, 2, \dots$$

It is clear that the product of two functions in $C_h(R)$ is again in $C_h(R)$, and likewise the limit of a U -convergent sequence of functions in $C_h(R)$. If $f(z)$ is the limit of a U -convergent sequence of finite products $f_n(z; h)$ one can write, in a disc about 0 which is free of zeros of $f(z)$,

$$(3.5) \quad f(z) = \exp(a_{h+1}z^{h+1} + a_{h+2}z^{h+2} + \dots),$$

where

$$(3.6) \quad a_j = \lim_{n \rightarrow \infty} s_{nj}, \quad j = h+1, h+2, \dots$$

The existence of the limits (3.6) by itself is not sufficient to imply U -convergence of a sequence $\{f_n(z; h)\}$. Something has to be added, as in the following lemma.

LEMMA 3.1 (CF. LINDWART AND PÓLYA [5]; OBRCHKOFF [6]). *Let the $f_n(z; h)$, $n = 1, 2, \dots$ be finite products (3.2) such that the associated sequences $\{s_{nj}\}$ (3.4) tend to limits a_j as $n \rightarrow \infty$, $j = h + 1, h + 2, \dots$. Suppose furthermore that there are an integer $t > h$ and a number M such that*

$$(3.7) \quad \sum_p |z_{np}|^{-t} \leq M, \quad n = 1, 2, \dots$$

Then the sequence $\{f_n(z; h)\}$ is U -convergent to an entire function of index $\leq t$.

There are simple geometrical conditions which imply (3.7):

LEMMA 3.2 (CF. LINDWART AND PÓLYA [5]; KOREVAAR [1]). *If the k th powers z_{np}^k of the zeros of the $f_n(z; h)$ belong to a fixed angle with vertex 0 and opening $< \pi$, and the numbers s_{nk} , $n = 1, 2, \dots$, form a bounded sequence, then (3.7) holds with $t = k$ and a suitable constant M . If the k th powers z_{np}^k all belong to a fixed half-plane $\alpha \leq \arg z \leq \alpha + \pi$, and the numbers s_{nk} and $s_{n, 2k}$, $n = 1, 2, \dots$, form bounded sequences, then there are constants A and B such that*

$$(3.8) \quad \sum_p |\operatorname{Im}(z_{np}^{-k} e^{i\alpha})| \leq A, \quad \sum_p |z_{np}|^{-2k} \leq B, \quad n = 1, 2, \dots$$

Here the requirement that the angle ($< \pi$ or $= \pi$) have vertex 0 may be dropped if it is known beforehand that the sequence $\{f_n(z; h)\}$ is U -convergent.

The following lemma helps one determine the zero free functions in $C_h(R)$.

LEMMA 3.3. *Let $z_n = r_n e^{i\theta_n}$, where $r_n \rightarrow \infty$ and $\theta_n \rightarrow \theta$. Let $\lambda > 0$, s a positive integer, and let ν_n be the integral part of $\lambda s r_n^s$. Then as $n \rightarrow \infty$,*

$$(3.9) \quad \{(1 - z/z_n) \exp(z/z_n + \dots + z^{s-1}/(s-1)z_n^{s-1})\}^{\nu_n} \rightarrow \exp(-\lambda e^{-is\theta} z^s)$$

in the sense of U -convergence.

The preceding lemmas may be used to obtain results on the class $C_h(R)$ in terms of asymptotic directions (cf. §1). We omit the proofs because they are very similar to those for the corresponding results on $C(R)$ (cf. Korevaar [1]).

THEOREM 3.4. *Let R be a set whose asymptotic directions are such that $A^{h+1}(R), \dots, A^{k-1}(R)$ (where $k \geq h+1$) do not lie in a half-plane. We denote the convex hull of $A^k(R)$ by S .*

(i) *Suppose that*

$$(3.10) \quad \exp(a_{h+1}z^{h+1} + \dots + a_k z^k + \dots)$$

represents a zero free entire function in $C_h(R)$. Then $a_k \in -\bar{S}$, and if S is an angle $< \pi$ moreover $a_j = 0$ for all $j > k$. On the other hand, if $a_k \in -\bar{S}$ then

$$(3.11) \quad \exp(a_{h+1}z^{h+1} + \dots + a_k z^k)$$

belongs to $C_h(R)$ for every choice of the complex numbers $a_{h+1}, \dots, a_{k-1}$.

(ii) Suppose that R^k belongs to an angle $< \pi$. Then $C_h(R)$ has index k , that is, the maximum of the indices of the functions in $C_h(R)$ is k . The class $C_h(R)$ consists of the entire functions which can be written in the form

$$(3.12) \quad \exp(b_{h+1}z^{h+1} + \dots + b_k z^k) \\ \cdot \prod_p (1 - z/z_p) \exp(z/z_p + \dots + z^{k-1}/(k-1)z_p^{k-1}),$$

where $z_p \in R$, $\sum_p |z_p|^{-k}$ converges, and $b_k \in -\bar{S}$.

(iii) Suppose that R^k belongs to the half-plane $\alpha \leq \arg z \leq \alpha + \pi$, but not to an angle $< \pi$. Then $C_h(R)$ has index q with $k < q \leq 2k$. All functions in $C_h(R)$ can be written in the form

$$(3.13) \quad \exp(b_{h+1}z^{h+1} + \dots + b_k z^k + \dots + b_q z^q) \\ \cdot \prod_p (1 - z/z_p) \exp(z/z_p + \dots + z^{q-1}/(q-1)z_p^{q-1}),$$

where $z_p \in R$, $\sum_p |z_p|^{-q}$ converges, and

$$(3.14) \quad b_k + \frac{1}{k} \sum_p z_p^{-k} \in -\bar{S},$$

in the sense that the series

$$(3.15) \quad \operatorname{Im}(b_k e^{i\alpha}) + \frac{1}{k} \sum_p \operatorname{Im}(z_p^{-k} e^{i\alpha})$$

converges to a sum

$$(3.16) \quad \sigma \begin{cases} \geq 0 & \text{if } S \text{ is the half-plane } \alpha \leq \arg z \leq \alpha + \pi, \\ = 0 & \text{if } S \text{ is the line } \arg z = \alpha, \alpha + \pi. \end{cases}$$

4. Initial results on $C(R)$ for radial sets R . Regular radial sets will be disposed of in a few lines. With every singular radial set we associate the sets R_ν and S_ν and the integers k_ν defined in §2. It will be shown that $C(R)$ contains the classes $C_{k_\mu}(R_{\mu+1})$, $\mu = 1, \dots, \omega - 1$, and this result is used to prove one direction of Theorems 2.1 and 2.2.

LEMMA 4.1. *A radial set R is regular if and only if no power R^j , $j = 1, 2, \dots$, belongs to a half-plane.*

Proof. (i) Suppose that no power R^j belongs to a half-plane. Then the same is true for the sets $A^j(R)$ ($= R^j$), hence by §1 (i) the set R is regular. (That all zero free entire functions belong to $C(R)$ could also be derived from Theorem 3.4.)

(ii) Suppose that R^k belongs to a half-plane. Since R^k is a radial set the half-plane will have the form $\alpha \leq \arg z \leq \alpha + \pi$. It now follows from

Theorem 3.4 that all functions in $C(R)$ are of order $\leq 2k$, hence R cannot be regular.

LEMMA 4.2. *For singular radial sets R one has the following inclusion relations:*

$$(4.1) \quad C(R) \supset C_0(R_1) \supset C_{k_1}(R_2) \supset \cdots \supset C_{k_{\omega-1}}(R_{\omega}).$$

Proof. It is clear from §3 that $C_0(R_1) \subset C(R_1) = C(R)$. Now let $h = k_{\mu-1}$ and $k = k_{\mu}$, where $1 \leq \mu \leq \omega - 1$. We will show that

$$C_h(R_{\mu}) \supset C_k(R_{\mu+1}).$$

Let $f(z) \in C_k(R_{\mu+1})$. Then $f(z)$ is the limit of a U -convergent sequence of finite products

$$(4.2) \quad f_n(z; k) = \prod_p (1 - z/z_{np}) \exp(-s_{n1}z - \cdots - s_{nk}z^k), \quad n = 1, 2, \dots,$$

where the z_{np} belong to $R_{\mu+1}$ and the s_{nj} are given by (3.4). Clearly

$$(4.3) \quad f_n(z; k) = \exp(-s_{n,h+1}z^{h+1} - \cdots - s_{nk}z^k) f_n(z; h).$$

By its definition the point $-\bar{s}_{nk}$ belongs to the convex hull of $R_{\mu+1}^k$. However, by the definition of $R_{\mu+1}$ the set $R_{\mu+1}^k$ is a line through 0, the boundary of S_{μ} . It follows that the point $\bar{s}_{nk}$ also belongs to S_{μ} , hence $-s_{nk} \in -\bar{S}_{\mu}$. Applying the first part of Theorem 3.4 to R_{μ} one concludes that

$$\exp(-s_{n,h+1}z^{h+1} - \cdots - s_{nk}z^k)$$

belongs to $C_h(R_{\mu})$. Since $z_{np} \in R_{\mu+1} \subset R_{\mu}$ it is clear that the function $f_n(z; h)$ belongs to $C_h(R_{\mu})$. Thus the products $f_n(z; k)$ (4.3) belong to $C_k(R_{\mu})$, and likewise their U -limit $f(z)$.

THEOREM 4.3. *Suppose that R is a singular radial set.*

(i) *Let $a_1, \dots, a_N$, where $N = k_{\omega}$, be any N -tuple of complex numbers such that*

$$(4.4) \quad a_{k_{\nu}} \in -\bar{S}_{\nu}, \quad \nu = 1, \dots, \omega.$$

Then the zero free entire function

$$(4.5) \quad \exp(a_1z + \cdots + a_Nz^N)$$

belongs to $C(R)$.

(ii) *More generally, let $f(z)$ be any entire function which can be written in the form*

$$(4.6) \quad \exp(b_1z + \cdots + b_Nz^N) \prod_p (1 - z/z_p) \exp(z/z_p + \cdots + z^{N-1}/(N-1)z_p^{N-1})$$

where $N = k_{\omega}$, $z_p \in R$, $\sum_p |z_p|^{-N}$ converges, $b_N \in -\bar{S}_{\omega}$, and

$$(4.7) \quad b_{k_\nu} + \frac{1}{k_\nu} \sum_p z_p^{-k_\nu} \in -\bar{S}_\nu, \quad \nu = 1, \dots, \omega - 1,$$

in the sense explained in the statement of Theorem 2.2. Then $f(z) \in C(R)$.

Proof. (i) Theorem 3.4 shows that for $\nu = 1, \dots, \omega$ the function

$$(4.8) \quad \exp(a_{k_{\nu-1}+1} z^{k_{\nu-1}+1} + \dots + a_{k_\nu} z^{k_\nu})$$

will belong to the corresponding class $C_{k_{\nu-1}}(R_\nu)$. However, by Lemma 4.2, the classes $C_{k_{\nu-1}}(R_\nu)$ are part of $C(R)$, hence the functions (4.8) all belong to $C(R)$. Forming their product one concludes that the function (4.5) belongs to $C(R)$.

(ii) Let us assume for definiteness that the product in (4.6) is infinite. Then $f(z)$ is the U -limit of the finite products

$$(4.9) \quad \phi_n(z) = \exp(c_1 z + \dots + c_N z^N) \prod_{p \leq n} (1 - z/z_p),$$

where

$$(4.10) \quad c_j = c_j(n) = b_j + \frac{1}{j} \sum_{p \leq n} z_p^{-j}, \quad j = 1, \dots, N-1; \quad c_N = b_N.$$

Because of (4.7) the number c_{k_ν} will either belong to $-\bar{S}_\nu$ or be very close to it. Indeed, set

$$(4.11) \quad d_{k_\nu} = c_{k_\nu} + i e^{-i\alpha_\nu} \frac{1}{k_\nu} \sum_{p > n} \operatorname{Im}(z_p^{-k_\nu} e^{i\alpha_\nu}), \quad \nu = 1, \dots, \omega - 1$$

(compare the statement of Theorem 2.2). Then

$$(4.12) \quad d_{k_\nu} - c_{k_\nu} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and

$$d_{k_\nu} e^{i\alpha_\nu} = \rho_\nu + i\sigma_\nu$$

where ρ_ν and σ_ν are real and σ_ν is the sum of the series (2.6). It thus follows from (2.7) that

$$(4.13) \quad d_{k_\nu} \in -\bar{S}_\nu, \quad \nu = 1, \dots, \omega - 1.$$

Setting $d_j = c_j$ for $j \neq k_\nu$, $\nu = 1, \dots, \omega - 1$ we will have $d_j - c_j \rightarrow 0$ as $n \rightarrow \infty$ for $j = 1, \dots, N$, hence $f(z)$ is also the U -limit of the modified products

$$(4.14) \quad \psi_n(z) = \exp(d_1 z + \dots + d_N z^N) \prod_{p \leq n} (1 - z/z_p).$$

Using (4.13) and the fact that $d_N = b_N \in -\bar{S}_\omega$ we obtain from part (i) of the theorem that

$$\exp(d_1 z + \dots + d_N z^N)$$

belongs to $C(R)$. It follows that the functions $\psi_n(z)$, and hence their U -limit $f(z)$, belong to $C(R)$.

5. First reduction theorem. Proof of Theorem 2.1. In this section we complete our investigation of the zero free functions in $C(R)$ for radial sets R . The principal tool is a kind of converse to Lemma 4.2. It shows that for every zero free entire function in $C(R)$ certain related functions belong to the classes $C_{k_\mu}(R_{\mu+1})$.

THEOREM 5.1 (FIRST REDUCTION THEOREM). *Let R be a singular radial set. Set $h = k_{\mu-1}$, $k = k_\mu$, where $1 \leq \mu \leq \omega - 1$, and suppose that*

$$(5.1) \quad f(z) = \exp(a_{h+1}z^{h+1} + \dots + a_k z^k + \dots)$$

is a zero free entire function in $C_h(R_\mu)$. Then

$$(5.2) \quad g(z) = \exp(a_{k+1}z^{k+1} + a_{k+2}z^{k+2} + \dots)$$

is a zero free entire function in $C_k(R_{\mu+1})$.

Proof. Write $f(z)$ as the limit of a U -convergent sequence of finite products $f_n(z; h)$ as in (3.2), with the z_{np} in R_μ . Since $f(z)$ is free of zeros,

$$(5.3) \quad r_n = \min_p |z_{np}| \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

The s_{nj} (3.4) will tend to a_j as $n \rightarrow \infty$, $j = h+1, h+2, \dots$ (3.6).

We next use the geometry of the set R_μ . Writing $\alpha_\mu = \alpha$ it is known that R_μ^k belongs to the half-plane $\alpha \leq \arg z \leq \alpha + \pi$ (and not to an angle $< \pi$). Setting $z_{np} = r_{np} \exp(i\theta_{np})$ one observes that

$$(5.4) \quad \sin(k\theta_{np} - \alpha) \geq 0.$$

By Lemma 3.2 there are constants A and B such that

$$(5.5) \quad \sum_p r_{np}^{-k} \sin(k\theta_{np} - \alpha) \leq A, \quad \sum_p r_{np}^{-2k} \leq B, \quad n = 1, 2, \dots$$

We now rotate all zeros z_{np} of $f_n(z; h)$ to the (or a) nearest ray of $R_{\mu+1}$. Their new positions w_{np} are given by

$$(5.6) \quad w_{np} = r_{np} \exp\{i(\alpha + s\pi)/k\},$$

where the integer $s = s_{np}$ is chosen so as to minimize

$$|\theta_{np} - (\alpha + s\pi)/k|,$$

subject to the condition that $\arg z = (\alpha + s\pi)/k$ be a ray of $R_{\mu+1}$. (When there are two possibilities for w_{np} one makes an arbitrary choice.) In terms of the w_{np} define

$$(5.7) \quad t_{nj} = -\frac{1}{j} \sum_p w_{np}^{-j},$$

and

$$(5.8) \quad \begin{aligned} g_n(z; k) &= \prod_p (1 - z/w_{np}) \exp(z/w_{np} + \dots + z^k/kw_{np}^k) \\ &= \exp(t_{n,k+1}z^{k+1} + t_{n,k+2}z^{k+2} + \dots), \quad n = 1, 2, \dots \end{aligned}$$

Clearly $g_n(z; k) \in C_k(R_{\mu+1})$. It will be shown that the $g_n(z; k)$ are U -convergent to $g(z)$, and thus $g(z) \in C_k(R_{\mu+1})$.

We will need an estimate for $t_{nj} - s_{nj}$. Since for real x ,

$$|e^{ix} - 1| \leq |x|,$$

one has the initial inequality

$$(5.9) \quad \begin{aligned} |t_{nj} - s_{nj}| &= \frac{1}{j} \left| \sum_p r_{np}^{-j} [\exp\{ -ij(\alpha + s_{np}\pi)/k \} - \exp(-ij\theta_{np})] \right| \\ &\leq \sum_p r_{np}^{-j} \frac{1}{j} |\exp[ij\{\theta_{np} - (\alpha + s_{np}\pi)/k\}] - 1| \\ &\leq \sum_p r_{np}^{-j} |\theta_{np} - (\alpha + s_{np}\pi)/k|. \end{aligned}$$

It will next be shown that there is a constant C (depending only on the set R_μ) such that for all n and p

$$(5.10) \quad |\theta_{np} - (\alpha + s_{np}\pi)/k| \leq C \sin(k\theta_{np} - \alpha).$$

Note that the complement of R_μ is an open radial set. Let us look at the rays of the special form $\arg z = (\alpha + s\pi)/k$ which belong to this complement. There will be a positive number $\delta < \pi/2k$ such that all rays within an angle δ of a special ray likewise belong to the complement of R_μ . Now consider the rays $\arg z = \theta$ of R_μ such that

$$\min_s |\theta - (\alpha + s\pi)/k| < \delta.$$

By the definition of δ the minimizing rays $\arg z = (\alpha + s\pi)/k$ must belong to R_μ and hence to $R_{\mu+1}$. For the rays $\arg z = \theta$ considered,

$$\min_s |k\theta - (\alpha + s\pi)| < k\delta < \frac{1}{2}\pi,$$

hence if $\arg z = (\alpha + s_\theta\pi)/k$ is the ray of $R_{\mu+1}$ nearest $\arg z = \theta$,

$$|k\theta - (\alpha + s_\theta\pi)| < \frac{1}{2}\pi \sin(k\theta - \alpha).$$

For all other rays $\arg z = \theta$ in R_μ one has

$$k\delta \leq \min_s |k\theta - (\alpha + s\pi)| \leq \frac{1}{2}\pi,$$

hence $\sin(k\theta - \alpha) \geq \sin k\delta$. It follows that for those rays $\arg z = \theta$ (again denoting by $\arg z = (\alpha + s_\theta\pi)/k$ the nearest ray of $R_{\mu+1}$),

$$|k\theta - (\alpha + s_\theta\pi)| \leq k\pi \leq \frac{k\pi}{\sin k\delta} \sin(k\theta - \alpha).$$

This completes the proof of (5.10).

Combining (5.9) with (5.10), and using (5.5) and (5.3), one obtains the estimate

$$\begin{aligned} |t_{nj} - s_{nj}| &\leq C \sum_p r_{np}^{-j} \sin(k\theta_{np} - \alpha) \\ (5.11) \quad &\leq C \sum_p r_{np}^{k-j} r_{np}^{-k} \sin(k\theta_{np} - \alpha) \leq AC r_n^{k-j}, \quad j \geq k, \quad n = 1, 2, \dots. \end{aligned}$$

Since the s_{nj} tend to a_j as $n \rightarrow \infty$ it immediately follows that

$$(5.12) \quad t_{nj} \rightarrow a_j \quad \text{as } n \rightarrow \infty, \quad j = k+1, k+2, \dots.$$

The second inequality (5.5) and (5.12) enable us to apply Lemma 3.1 to the functions $g_n(z; k)$. The conclusion is that the $g_n(z; k)$ are U -convergent. The limit function $G(z)$ will have the form $\exp(b_{k+1}z^{k+1} + b_{k+2}z^{k+2} + \dots)$. Here $b_j = \lim t_{nj}$, hence by (5.12) $b_j = a_j$ and thus $G(z) = g(z)$. It follows that $g(z) \in C_k(R_{\mu+1})$.

We now have all the ingredients for the

Proof of Theorem 2.1. Let R be a singular radial set. Suppose that

$$(5.13) \quad \exp(a_1z + a_2z^2 + \dots)$$

represents a zero free entire function in $C(R)$. Repeated application of Theorem 5.1 then shows that for $\nu = 1, \dots, \omega$,

$$(5.14) \quad \exp(a_{k_{\nu-1}+1}z^{k_{\nu-1}+1} + \dots + a_{k_\nu}z^{k_\nu} + \dots)$$

represents a zero free entire function in $C_{k_{\nu-1}}(R_\nu)$. Thus by Theorem 3.4

$$(5.15) \quad a_{k_\nu} \in -\bar{S}_\nu, \quad \nu = 1, \dots, \omega.$$

For $\nu = \omega$ the set S_ν is an angle $< \pi$, hence Theorem 3.4 shows in addition that $a_j = 0$ for all $j > k_\omega$.

It follows that all zero free entire functions in $C(R)$ have indeed the form given by (2.2) and (2.3). The converse was obtained earlier as part (i) of Theorem 4.3.

6. Second reduction theorem. Proof of Theorem 2.2. Let R be a singular radial set, and let q be the index of $C(R)$. Since R^{k_1} belongs to a half-plane, $q \leq 2k_1$ (Theorem 3.4). On the other hand, by Theorem 4.3, $q \geq N = k_\omega$. By Lemma 4.2 the classes $C_{k_\mu}(R_{\mu+1})$ all have index $\leq q$.

We will prove another reduction theorem, more general than the one in §5, which shows that for every entire function in $C(R)$ certain related functions belong to the classes $C_{k_\mu}(R_{\mu+1})$. We start by describing the construction of these functions.

CONSTRUCTION 6.1. Let $h = k_{\mu-1}$, $k = k_\mu$, where $1 \leq \mu \leq \omega - 1$, and suppose that $f(z) \in C_h(R_\mu)$. Then $f(z)$ can be written in the form

$$(6.1) \quad f(z) = \exp(c_{h+1}z^{h+1} + \dots + c_q z^q) \\ \cdot \prod_p (1 - z/z_p) \exp(z/z_p + \dots + z^{q-1}/(q-1)z_p^{q-1}),$$

where q is the index of $C(R)$, $z_p \in R_\mu$ and $\sum_p |z_p|^{-q}$ converges. One can say a little more. The set R_μ^k belongs to the half-plane $\alpha \leq \arg z \leq \alpha + \pi$ where $\alpha = \alpha_\mu$. Thus by Theorem 3.4, writing $z_p = \exp(i\theta_p)$, the series

$$(6.2) \quad \sum_p r_p^{-k} \sin(k\theta_p - \alpha) = \sum_p -\operatorname{Im}(z_p^{-k} e^{i\alpha})$$

of non-negative terms converges.

We now rotate all zeros z_p of $f(z)$ to the (or a) nearest ray of $R_{\mu+1}$. Their new positions w_p are given by

$$(6.3) \quad w_p = r_p \exp\{i(\alpha + s_p \pi)/k\},$$

where the integer s_p is chosen so as to minimize $|\theta_p - (\alpha + s_p \pi)/k|$, subject to the condition that the ray $\arg z = (\alpha + s_p \pi)/k$ belong to $R_{\mu+1}$. For some points z_p there may be two candidates for w_p . In this case we let the choice of w_p depend on a given sequence $\{f_n(z; h)\}$ which is U -convergent to $f(z)$, as explained in the proof of Theorem 6.2 below.

In terms of the numbers w_p define

$$(6.4) \quad d_j = c_j - \frac{1}{j} \sum_p (w_p^{-j} - z_p^{-j}), \quad k < j < q; \quad d_q = c_q.$$

The convergence of the series (6.2) and the argument used in the proof of Theorem 5.1 show that the series in (6.4) are absolutely convergent (cf. (5.9)-(5.11)).

Finally set

$$(6.5) \quad g(z) = \exp(d_{k+1}z^{k+1} + \dots + d_q z^q) \\ \cdot \prod_p (1 - z/w_p) \exp(z/w_p + \dots + z^{q-1}/(q-1)w_p^{q-1});$$

it is clear that $\sum_p |w_p|^{-q}$ converges.

THEOREM 6.2 (SECOND REDUCTION THEOREM). *Let $f(z) \in C_k(R_\mu)$, and let $g(z)$ be the function associated with $f(z)$ by Construction 6.1. Then $g(z) \in C_k(R_{\mu+1})$. If $f(z)$ has its index equal to q then so does $g(z)$.*

Proof. As in the proof of Theorem 5.1 we write $f(z)$ as the limit of a U -convergent sequence of functions $f_n(z; h)$ of the form (3.2), with the $z_{np} \in R_\mu$. For each fixed n the z_{np} will be numbered in such a way that $z_{np} \rightarrow z_p$ as $n \rightarrow \infty$, $p = 1, 2, \dots$. Writing $z_{np} = r_{np} \exp(i\theta_{np})$ one again has (5.4) and (5.5). For the numbers s_{nj} defined by (3.4) one will have, as $n \rightarrow \infty$,

$$(6.6) \quad s_{nj} \rightarrow \begin{cases} c_j & \text{for } h < j < q, \\ c_q - \frac{1}{q} \sum_p z_p^{-q} & \text{for } j = q, \\ -\frac{1}{j} \sum_p z_p^{-j} & \text{for } j > q. \end{cases}$$

Next introduce numbers w_{np} as in the proof of Theorem 5.1. For every p for which there was only one candidate w_p in Construction 6.1 one will have $w_{np} \rightarrow w_p$ as $n \rightarrow \infty$. However, for a p for which there were two candidates w'_p and w''_p it is known only that every w_{np} (with n large) will be close to either w'_p or w''_p . One can resolve this difficulty as follows. Pick a subsequence of n 's such that the corresponding sequence $\{w_{n1}\}$ converges, and denote the limit (which is either w_1 , or w'_1 , or w''_1) by w_1 . Next pick a subsequence of the subsequence such that the corresponding sequence $\{w_{n2}\}$ converges, etc. For the diagonal sequence of n 's, $w_{np} \rightarrow w_p$ for every p . The corresponding sequence of functions $f_n(z; h)$ will again be denoted by $\{f_n(z; h)\}$.

One finally defines numbers t_{nj} and a sequence $\{g_n(z; k)\}$ of functions in $C_k(R_{\mu+1})$ as in the proof of Theorem 5.1. *It will be shown that the $g_n(z; k)$ (5.8) are U -convergent to $g(z)$ (6.5), and thus $g(z) \in C_k(R_{\mu+1})$.*

We first prove that for $j > k$

$$(6.7) \quad t_{nj} - s_{nj} = -\frac{1}{j} \sum_p (w_{np}^{-j} - z_{np}^{-j}) \rightarrow -\frac{1}{j} \sum_p (w_p^{-j} - z_p^{-j})$$

as $n \rightarrow \infty$. Taking $r \neq$ all r_p it is clear that

$$\sum_{r_{np} < r} (w_{np}^{-j} - z_{np}^{-j}) \rightarrow \sum_{r_p < r} (w_p^{-j} - z_p^{-j})$$

as $n \rightarrow \infty$. Furthermore, by the argument used in the proof of Theorem 5.1 (cf. (5.11)),

$$\left| \frac{1}{j} \sum_{r_{np} \geq r} (w_{np}^{-j} - z_{np}^{-j}) \right| \leq A C r^{k-j}, \quad n = 1, 2, \dots$$

The same inequality holds with w_{np} , z_{np} and r_{np} replaced by w_p , z_p and r_p . Taking r large one obtains (6.7).

Combining (6.6) and (6.7) one finds that as $n \rightarrow \infty$,

$$(6.8) \quad t_{nj} \rightarrow \begin{cases} d_j & \text{for } k < j < q, \\ d_q - \frac{1}{q} \sum_p w_p^{-q} & \text{for } j = q, \\ -\frac{1}{j} \sum_p w_p^{-j} & \text{for } j > q, \end{cases}$$

where the d_j are given by (6.4). By the second inequality (5.5)

$$(6.9) \quad \sum_p |w_{np}|^{-2k} \leq B, \quad n = 1, 2, \dots$$

Thus by Lemma 3.1 our functions $g_n(z; k)$ are U -convergent. For $|z| < \min r_p$ the limit function $G(z)$ can be written in the form

$$\exp(e_{k+1}z^{k+1} + e_{k+2}z^{k+2} + \dots).$$

Here $e_j = \lim t_{nj}$, hence by (6.8) and (6.5), $G(z) = g(z)$. It follows that $g(z) \in C_k(R_{\mu+1})$.

Finally, suppose that $f(z)$ has its index equal to q , the index of $C(R)$. If $q > 1$ then either $c_q \neq 0$ or the series $\sum_p r_p^{-q+1}$ diverges (or both). Hence in that case $g(z)$ also has its index equal to q . If $q = 1$ there is nothing to prove.

We can now give the

Proof of Theorem 2.2. Let R be a singular radial set, and let q be the index of the class $C(R)$. Repeated application of Theorem 6.2 shows that the classes $C_{k_\mu}(R_{\mu+1})$ also have index q . However, $R_\omega^{k_\omega}$ is contained in an angle $< \pi$, hence by part (ii) of Theorem 3.4 the class $C_{k_{\omega-1}}(R_\omega)$ has index k_ω . It follows that $q = k_\omega = N$.

Suppose now that $f(z)$ is any function in $C_0(R)$. Then $f(z)$ can be written in the form

$$(6.10) \quad \exp(b_1 z + \dots + b_N z^N) \prod_p (1 - z/z_p) \exp(z/z_p + \dots + z^{N-1}/(N-1)z_p^{N-1})$$

where $z_p \in R$ and $\sum_p |z_p|^{-N}$ converges. Repeated application of Theorem 6.2 shows that $C_{k_{\nu-1}}(R_\nu)$ contains an entire function of the form

$$(6.11) \quad \exp(b_{k_{\nu-1}}^{(\nu)} z^{k_{\nu-1}+1} + \dots + b_N^{(\nu)} z^N) \cdot \prod_p (1 - z/z_p^{(\nu)}) \exp(z/z_p^{(\nu)} + \dots + z^{N-1}/(N-1)z_p^{(\nu)N-1}),$$

where $|z_p^{(\nu)}| = |z_p|$, and

$$\begin{aligned}
 b_j^{(\nu)} &= b_j^{(\nu-1)} - \frac{1}{j} \sum_p \{ [z_p^{(\nu)}]^{-j} - [z_p^{(\nu-1)}]^{-j} \} = \dots \\
 (6.12) \quad &= b_j - \frac{1}{j} \sum_p \{ [z_p^{(\nu)}]^{-j} - z_p^{-j} \}, \quad k_{\nu-1} < j < N, \\
 b_N^{(\nu)} &= b_N.
 \end{aligned}$$

For $\nu = 1, \dots, \omega - 1$ one can apply part (iii) of Theorem 3.4 to the function in (6.11). Thus one finds that the series

$$(6.13) \quad \operatorname{Im}(b_{k_\nu}^{(\nu)} e^{i\alpha_\nu}) + \frac{1}{k_\nu} \sum_p \operatorname{Im}([z_p^{(\nu)}]^{-k_\nu} e^{i\alpha_\nu})$$

converges to a sum σ_ν which is ≥ 0 if S_ν is the half-plane $\alpha_\nu \leq \arg z \leq \alpha_\nu + \pi$, and $= 0$ if S_ν is the line $\arg z = \alpha_\nu, \alpha_\nu + \pi$. Combining (6.13) and (6.12) (with $j = k_\nu$) one concludes that the series

$$(6.14) \quad \operatorname{Im}(b_{k_\nu} e^{i\alpha_\nu}) + \frac{1}{k_\nu} \sum_p \operatorname{Im}(z_p^{-k_\nu} e^{i\alpha_\nu})$$

also converges to the sum σ_ν .

For $\nu = \omega$ one uses part (ii) of Theorem 3.4 to show that $b_N \in -\bar{S}_\omega$.

It follows that all functions in $C(R)$ have indeed the form given in Theorem 2.2. The converse was obtained earlier as part (ii) of Theorem 4.3.

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